UNIFORM CONVERGENCE OF FOURIER SERIES

1. Introduction. It is well known that the Fourier series of a continuous function is not necessarily uniformly convergent in an interval of continuity of the sum-function. However, if, in addition to being continuous, the function is also of bounded variation then it is known that the Fourier series is uniformly convergent in the interval of periodicity. This is the so-called Jordan criterion for uniform convergence.

A second criterion for uniform convergence is the Dini-Lipschitz condition. If

$$|f(x+t)-f(x)| < \frac{K}{\left(\log \frac{1}{|t|}\right)^{1+\alpha}} \qquad \alpha,\, K > 0$$

then the Fourier series of $f(x)$ converges uniformly. This condition may be slightly improved. The essential point to be observed here, as far as the results of this paper are concerned, is that the power of $\log 1/|t|$ exceeds unity.

It was proved by H. E. Bray[1] that continuous functions of *écart fini*[2] have uniformly convergent Fourier series. A function i(x), summable for $0 \leqq x \leqq 2\pi$ is said to be of *écart fini* in this nterval if the integrals

$$\int_a^b f(x) n \sin nx dx \qquad \int_a^b f(x) n \cos nx dx$$

are bounded independently of the integer n and the numbers a, b; where $0 \leqq a < b \leqq 2\pi$. This result contains the above-mentioned Jordan criterion since every function of bounded variation is a function of *écart fini*.

Bray also showed[3] that if the Fourier coefficients of a bounded and measurable function are $O(1/n \log n)$ the function is a function of *écart fini*.

The present paper has a two-fold purpose. First, to define a

[1] Am. Jour. Math., Jan. 1929, p. 149 et seq.
[2] Hadamard. Jour. de Liouville, Ser. 4, v. 8, p. 154.
[3] Comptes Rendus, t 190, p. 1371.

class of functions wider than the class of functions of *écart fini* and to obtain sufficient conditions for the existence of functions of this class. Second, to show that if continuous functions of this new class have moduli of continuity satisfying a slight restriction—less restrictive, in fact, than the Dini-Lipschitz condition—then their Fourier series converge uniformly. These functions do not satisfy the classical convergence criteria. An interesting example of a function of this kind, for which the Dini-Lipschitz criterion is not satisfied, is exhibited.

2. Functions of Class $\phi(n)$. A summable function $f(x)$ of period 2π will be said to be of class $\phi(n)$ in the interval $0 \leqq x \leqq 2\pi$ if

$$\phi(n) \int_a^b f(x+t) \cos ntdt = O(1) \tag{1}$$

uniformly for all x, n, a, b with $b-a \leqq 2\pi$. If, in particular, $\phi(n) = n$, the function $f(x)$ is a function of *écart fini*.

THEOREM 1. *If the Fourier coefficients of a bounded and measurable function $f(x)$ of period 2π are of the order $1/\psi(n)$, where $n \leqq \psi(n) \leqq n \log n$, and if the condition*

$$\frac{d\psi(t)}{dt} > k\,\frac{\psi(t)}{t} \qquad k > 0^{*} \tag{2}$$

is satisfied, then $f(x)$ is of class $\phi(n) = \psi(n)/\log n$.

It should be noted that we must have $\phi(n) = O(n)$ except in the trivial case where $f(x)$ is a null function. For suppose that $\phi(n)/n$ were unbounded with $f(x)$ not a null function. We consider the expression

$$\phi(n) \int_0^{\pi/2n} f(x+t) \cos ntdt.$$

An integration by parts gives

$$\int_0^{\pi/2n} f(x+t) \cos ntdt = n \int_0^{\pi/2n} [F(x+t) - F(x)] \sin ntdt$$

where

* This condition assures us that $\psi(t)$ is an increasing function.

$$F(x+t) = F(x) + \int_0^t f(x+y)dy.$$

If we suppose that $f(x)$ is not a null function (we may assume that $f(x) > 0$), then for some x we shall have

$$\frac{F(x+t) - F(x)}{t} > q > 0$$

for $|t|$ sufficiently small. Hence

$$n \int_0^{\pi/2n} [F(x+t) - F(x)] \sin ntdt > n \int_0^{\pi/2n} qt \sin ntdt = q/n$$

for n sufficiently large. Therefore

$$(3) \qquad \phi(n) \int_0^{\pi/2n} f(x+t) \cos ntdt > \frac{\phi(n)}{n} q.$$

But as $n \to \infty$, $\phi(n)/n \to \infty$. It follows that the expression (3) is unbounded. This is in contradiction with the definition of functions of this class, for $f(x)$ is summable and therefore expression (3) is uniformly bounded.

In order to prove the theorem we make use of two lemmas. The first concerns the expression

$$E(x, m, \delta) = \sum_{r=1}^{m} (-1)^r [F(x + r\delta) - F(x + (r-1)\delta)]$$

with m a positive integer and δ any positive number such that $m\delta \leqq 2\pi$.

LEMMA 1. *If $\phi(n) = O(n)$, and if $f(x)$ is bounded and measurable and is such that $E(x, m, \delta) = O(1/\phi(1/\delta))$, then*

$$\int_a^b f(x+t) \cos ntdt = O(1/\phi(n)).$$

Since $f(x)$ is bounded, a part of the interval of length $O(1/n)$ may be disregarded in considering the boundedness of the expression $|\phi(n)\int_a^b f(x+t) \cos ntdt|$. But $\phi(n) \leqq O(n)$ so we may discard a part of $[a, b]$ of length $O(1/n)$. The integral to be considered is the difference of two integrals of the form

$\int_0^a f(x+t)$ cos $ntdt$ so it suffices to consider this integral. We have, after letting $a=m\pi/n$ and integrating by parts,

$$\int_0^{m\pi/n} f(x+t)\cos ntdt = (-1)^m F\left(x+m\frac{\pi}{n}\right) - F(x)$$
$$+\int_0^{m\pi/n} F(x+t)n\sin ntdt.$$

But

$$\int_0^{m\pi/n} F(x+t)n\sin ntdt$$
$$=\frac{1}{2}\int_0^{\pi/n}[F(x+t)+(-1)^{m-1}F(x+t+(m-1)\pi/n)]n\sin ntdt$$
$$+\frac{1}{2}\int_0^{\pi/n} E(x+t,\, m-1,\, \pi/n)n\sin ntdt.$$

Since $E=O(1/\phi(n))$, the last integral is $O(1/\phi(n))$. We therefore have

$$\int_0^{m\pi/n} f(x+t)\cos ntdt$$
$$=\frac{1}{2}\int_0^{\pi/n}[F(x+t)+(-1)^{m-1}F(x+t+(m-1)\pi/n)]n\sin ntdt$$
$$-F(x)+(-1)^m F(x+m\pi/n)+O(1/\phi(n)).$$

This can be written as

$$\int_0^{m\pi/n} f(x+t)\cos ntdt$$
$$=\frac{1}{2}\int_0^{\pi/n}[F(x+t)-F(x)]n\sin ntdt$$
$$+\frac{1}{2}(-1)^{m-1}\int_0^{\pi/n}\left[F\left(x+t+(m-1)\frac{\pi}{n}\right)\right. \tag{4}$$
$$\left.-F\left(x+\frac{m\pi}{n}\right)\right]n\sin ntdt+O(1/\phi(n)).$$

But

$$F(x+t) - F(x) = O\left(1/\phi\left(\frac{1}{t}\right)\right) = O(1/\phi(n))$$

Since $O \leqq t \leqq \pi/n$. We may also write

$$\begin{aligned} F(x + t + (m-1)\pi/n) - F(x + \pi m/n) \\ = F(x + (m-1)\pi/n + t) - F(x + (m-1)\pi/n) \\ - [F(x + (m-1)\pi/n + \pi/n) - F(x + (m-1)\pi/n)], \end{aligned}$$

and thus

$$F(x + t + (m-1)\pi/n) - F(x + m\pi/n) = O(1/\phi(n)).$$

Hence each term in the right member of (4) is of the order $1/\phi(n)$, and the lemma is proved.

The trapezoidal sum is used in the proof of the second lemma. This sum is defined on the interval $a \leqq x \leqq b$ for the function $F(x)$ as follows:

$$\begin{aligned} T(a, b, h) = (h/2)[F(a) + 2F(a+h) + 2F(a+2h) + \cdots \\ + 2F(a+(m-1)h) + F(a+mh)] \end{aligned}$$

where $a + mh = b$.

LEMMA 2. *If* $T(a, b, h) - \int_a^b F(x)dx = O(h/\phi(1/h))$ *uniformly in* $a \leqq x \leqq b$, *then* $E(x, m, h) = O(1/\phi(1/h))$.

We have

$$T(a, b, h) - \int_a^b F(x)dx = O(h/\phi(1/h))$$

$$T(a, b, h/2) - \int_a^b F(x)dx = O([h/2]/\phi(2/h)) = O(h/\phi(1/h)).$$

Thus $T(a, b, h) - T(a, b, h/2) = O(h/\phi(1/h)$, and

$$\begin{aligned} F(a) - 2F(a + h/2) + \cdots + (-1)^{2m}F(a+mh) \\ = O(1/\phi(1/h)). \end{aligned}$$

This is the desired result, and the lemma is proved.

We are now able to prove Theorem 1. From condition (2) we obtain the condition

$$\int_n^\infty \frac{dt}{t\psi(t)} = O(1/\psi(n)). \tag{5}$$

For since $\psi'(t)/\psi(t) > k/t$ we have

$$\int_n^\infty \frac{dt}{t\psi(t)} < \frac{1}{k}\int_n^\infty \frac{\psi'(t)}{(\psi(t))^2}\,dt = \frac{1}{k\psi(n)}\,.$$

We may suppose that, almost anywhere,

$$f(x) = \sum_{r=1}^\infty (a_r \cos rx + b_r \sin rx)$$

since it is clear that when (1) is satisfied for $f(x)$ it is satisfied for $f(x) - a_0/2$. There is no loss of generality in supposing also that

$$F(x) = \sum_{r=1}^\infty \frac{a_r \sin rx - b_r \cos rx}{r}\,.$$

In order to prove the theorem it will suffice to show that

$$T(a, b, h) - \int_a^b F(x)dx = O\left(\frac{h \log (1/h)}{\psi(1/h)}\right).$$

For the sake of brevity we shall write $\phi(t) = \psi(t)/\log t$. We thus wish to show that

$$T(a, b, h) - \int_a^b F(x)dx = O(h/\phi(1/h)).$$

If we set $b = a + mh$ with $0 < b - a \leqq 2\pi$, we obtain

$$T(a, b, h) = \frac{h}{2}\,[F(a) + 2F(a+h) + \cdots + F(a+mh)]$$

$$= \frac{h}{2}\left[\sum_{i=0}^{m-1} F(a+ih) + \sum_{i=1}^{m} F(a+ih)\right] \tag{6}$$

$$= \frac{h}{2} \sum_{r=1}^{\infty} \left[\sum_{i=0}^{m-1} + \sum_{1}^{m} \frac{a_r \sin r(a+ih) - b_r \cos r(a+ih)}{r} \right].$$

But

$$\sum_{i=0}^{m-1} + \sum_{i=1}^{m} \frac{a_r \sin r(a+ih)}{r} = \frac{a_r}{r} (\cos ra - \cos r(a+mh)) \operatorname{ctn} \frac{rh}{2}$$

$$\sum_{i=0}^{m-1} + \sum_{i=1}^{m} \frac{b_r \cos r(a+ih)}{r} = \frac{b_r}{r} (-\sin ra + \sin r(a+mh)) \operatorname{ctn} \frac{rh}{2} \cdot$$

Thus we obtain

$$T(a, b, h) = \frac{h}{2} \sum_{r=1}^{\infty}{}' \frac{A_r(a) - A_r(a+mh)}{r} \operatorname{ctn} \frac{rh}{2} + \sigma \tag{7}$$

where $A_n(x) = a_n \cos nx + b_n \sin nx$. The accent indicates that in the summation r runs through all positive integer values except those for which $rh/2$ is a multiple of π. These will be considered separately, their sum being represented by σ.

We also have

$$\int_a^{a+mh} F(x)dx = - \sum_{r=1}^{\infty} \frac{A_r(x)}{r^2} \Bigg]_a^{a+mh}$$

$$= - \sum_{r=1}^{\infty}{}' \frac{A_r(a+mh) - A_r(a)}{r^2}$$

with the same interpretation of the prime over the summation

$$\begin{aligned} T(a, b, h) &- \int_a^b F(x)dx \\ &= \frac{h}{2} \sum_{1}^{\infty}{}' \frac{A_r(a) - A_r(a+mh)}{r^2} \operatorname{ctn} \frac{rh}{2} + \sigma \\ &\quad - \sum_{1}^{\infty}{}' \frac{A_r(a) - A_r(a+mh)}{r^2} \\ &= \sum_{r=1}^{\infty}{}' \frac{A_r(a) - A_r(a+mh)}{r^2} \left(\frac{rh}{2} \operatorname{ctn} \frac{rh}{2} - 1 \right) + \sigma. \end{aligned} \tag{8}$$

We wish to show that (8) is $O(h/\phi(1/h))$, that is

$$\sum_{r=1}^{\infty}{}' \frac{A_r(a) - A_r(a+mh)}{r^2 h} \phi(1/h) \left(\frac{rh}{2} \operatorname{ctn} \frac{rh}{2} - 1\right) \tag{9}$$

$$+ \sigma\phi(1/h)/h = O(1).$$

We consider first the terms of σ of (7) for which $rh/2 = k\pi$, $k = 1, 2, 3, \cdots$. We replace h by π/n and consider the terms for which $r = 2kn$. If the right number of (6) is divided by $h/\phi(1/h) = \pi/n\phi(n/\pi)$ we obtain

$$\frac{\pi}{2n} \sum_{k=1}^{\infty} \left\{ \sum_{i=0}^{m-1} + \sum_{1}^{m} \left[a_{2kn} \sin 2kn(a+ih)\right.\right.$$

$$\left. - b_{2kn} \cos 2kn(a+ih)\right] \frac{n\phi(n/\pi)}{\pi}$$

which is equal to

$$\frac{\phi\left(\dfrac{n}{\pi}\right)}{2} \sum_{k=1}^{\infty} \frac{a_{2kn} \sin 2akn - b_{2kn} \cos 2akn}{2kn} 2m. \tag{10}$$

Since $mh = m\pi/n \leqq 2\pi$ we have $m = O(n)$. The expression (10) does not exceed in absolute value a quantity of the order

$$m\phi(n) \sum_{k=1}^{\infty} \frac{1}{kn\psi(2kn)}. \tag{11}$$

But

$$\sum_{k=1}^{\infty} \frac{1}{k\psi(2kn)} \leqq \int_1^{\infty} \frac{dt}{t\psi(nt)} = \int_n^{\infty} \frac{dt}{t\psi(t)}$$

$$= O(1/\psi(n)) \qquad \text{from (5).}$$

It follows that (11) is of order $\phi(n)/\psi(n) = 1/\log n$ which is $o(1)$. This is a sharper result than we need.

If we denote the expression (9) by $P(a, b, h)$ and replace h by π/n we may write

$P(a, b, h)$

$$= \sum_{r=1}^{[n]} \frac{A_r(a)-A_r(a+m\pi/n)}{r^2\pi}\,\phi\left(\frac{n}{\pi}\right)\left(\frac{r\pi}{2n}\operatorname{ctn}\frac{r\pi}{2n}-1\right)n$$

(12)
$$+ \sum_{[n]+1}^{\infty}{}' \frac{A_r(a)-A_r\left(a+m\dfrac{\pi}{n}\right)}{2r}\,\phi\left(\frac{n}{\pi}\right)\operatorname{ctn}\frac{r\pi}{2n}$$

$$- \sum_{[n]+1}^{\infty} \frac{A_r(a)-A_r\left(a+m\dfrac{\pi}{n}\right)}{r^2\pi}\,n\phi\left(\frac{n}{\pi}\right) + 0(1).$$

The third sum here is in absolute value not greater than

$$\sum_{r=[n]+1}^{\infty} \left| \frac{A_r(a)-A_r\left(a+m\dfrac{\pi}{n}\right)}{r^2\pi}\,n\phi\left(\frac{n}{\pi}\right) \right| < \sum_{[n]+1}^{\infty} \left| \frac{K\phi\left(\dfrac{n}{\pi}\right)n}{r^2\pi\psi(r)} \right|$$

where K is some positive constant. The above expression does not exceed a quantity of the order

$$n\phi(n) \sum_{[n]+1}^{\infty} \left|\frac{1}{r^2\psi(r)}\right| < \frac{\phi(n)}{\psi(n)} = o(1), \qquad by\ (5).$$

The second sum of (12) does not exceed in absolute value a quantity of the order

(13)
$$\sum_{r=[n]+1}^{\infty}{}' \left|\frac{\phi(n)\operatorname{ctn} r\pi/2n}{r\psi(r)}\right| = \sum_{[n]+1}^{2[n]-1} + \sum_{2[n]+1}^{\infty}{}' \left|\frac{\phi(n)\operatorname{ctn} r\pi/2n}{r\psi(r)}\right|.$$

The first sum of (13) does not exceed

$$\frac{\phi(n)}{n\psi(n)} \sum_{[n]+1}^{2[n]-1} \left|\operatorname{ctn}\frac{r\pi}{2n}\right| = \frac{1}{n\log n}\sum_{1}^{n-1}\left|\operatorname{ctn}\left(\frac{r\pi}{2n}+\frac{\pi}{2}\right)\right|$$
$$= \frac{1}{n\log n}\sum_{1}^{n-1}\left|\tan\frac{r\pi}{2n}\right|.$$

But

$$\sum_{r=1}^{n-1}\left|\tan\frac{r\pi}{2n}\right| < \int_1^{n-1} \tan\frac{\pi x}{2n}\,dx = -\frac{2n}{\pi}\log\tan\frac{\pi}{2n}\,.$$

Since $\tan(\pi/2n)=O(1/n)$ it follows that

$$\frac{2n}{\pi}\log\tan(\pi/2n) = O(n\log n).$$

Hence the first sum of (13) has no greater order than

$$[1/(n\log n)]O(n\log n) = O(1).$$

For the second sum in (13) we have

$$\sum_{r=2n+1}^{\infty}{}' \left|\frac{\phi(n)\operatorname{ctn}\dfrac{r\pi}{2n}}{r\psi(r)}\right| \leqq \phi(n)\sum_{k=1}^{\infty}\sum_{r=2nk+1}^{2n(k+1)-1}\left|\frac{\operatorname{ctn}\dfrac{r\pi}{2n}}{r\psi(r)}\right|. \tag{14}$$

The second sum here gives

$$\begin{aligned}\sum_{r=2nk+1}^{2n(k+1)-1}\left|\frac{\operatorname{ctn}\dfrac{r\pi}{2n}}{r\psi(r)}\right| &< \frac{1}{2nk\psi(2nk)}\sum_{2nk+1}^{2n(k+1)-1}\left|\operatorname{ctn}\frac{r\pi}{2n}\right|\\ &< \frac{1}{2nk\psi(2nk)}\sum_{r=1}^{2n-1}\left|\operatorname{ctn}\frac{r+2nk}{2n}\pi\right|\\ &< \frac{1}{2nk\psi(2nk)}\sum_{r=1}^{2n-1}\left|\operatorname{ctn}\frac{r\pi}{2n}\right|.\end{aligned}$$

But

$$\sum_{r=1}^{2n-1}\left|\operatorname{ctn}\frac{r\pi}{2n}\right| = \sum_{r=1}^{n}+\sum_{n+1}^{2n-1}\left|\operatorname{ctn}\frac{r\pi}{2n}\right|$$

and

$$\sum_{r=1}^{n}\left|\operatorname{ctn}\frac{r\pi}{2n}\right| < \int_1^n \operatorname{ctn}\frac{\pi x}{2n}\,dx = \frac{2n}{\pi}\left|\log\sin\frac{\pi}{2n}\right| = O(n\log n).$$

Similarly

$$\sum_{r=n+1}^{2n-1}\left|\operatorname{ctn}\frac{r\pi}{2n}\right| < O(n\log n).$$

It therefore follows that (14) does not exceed a quantity of the order

$$\phi(n)\sum_{k=1}^{\infty}\frac{n\log n}{nk\psi(nk)}\leqq\phi(n)\log n\sum_{k=1}^{\infty}\frac{1}{k\psi(nk)}\,. \tag{15}$$

We have already seen that the last sum in (15) is not greater than a quantity of the order $1/\psi(n)$. It follows that (14) is of the order $\phi(n)\log n/\psi(n)=1$.

We have shown thus far that the second and third sums of (12) are $O(1)$. We must now show that

$$\sum_{r=1}^{n}\frac{A_r(a)-A_r(a+m\pi/n)}{r^2\pi}\,n\phi\left(\frac{n}{\pi}\right)\left(\frac{r\pi}{2n}\operatorname{ctn}\frac{r\pi}{2n}-1\right)=O(1).$$

The expression

$$\frac{1-\dfrac{r\pi}{2n}\operatorname{ctn}\dfrac{r\pi}{2n}}{r^2\pi}\,n\phi\left(\frac{n}{\pi}\right) \tag{16}$$

is bounded as $n\to\infty$ for fixed r. This is easily seen since from a well known formula*

$$\frac{1-x\operatorname{ctn}x}{x^2}=\frac{1}{3}+\frac{x^2}{45}+\frac{2x^4}{945}+\frac{x^6}{4725}+\cdots.$$

Moreover, (16) is an increasing function of r for each fixed n. Its greatest value occurs therefore when $r=n$; namely

$$\frac{\phi\left(\dfrac{n}{\pi}\right)}{n\pi}\leqq K/\pi^2,$$

where K is a positive constant.

If we use the second mean value theorems for sums, we obtain

$$\frac{K}{\pi^2}\sum_{k'\leqq n}\left[A_r\left(a+m\frac{\pi}{n}\right)-A_r(a)\right]$$

* B. O. Peirce, Tables.

$$\leqq \sum_1^n \left[A_r\left(a + m\frac{\pi}{n}\right) - A_r(a)\right] \frac{1 - \dfrac{r\pi}{2n}\operatorname{ctn}\dfrac{r\pi}{2n}}{r^2\pi}\, n\phi\left(\frac{n}{\pi}\right)$$

$$\leqq \frac{K}{\pi^2} \sum_{k'' \leqq n}^{n} \left[A_r\left(a + m\frac{\pi}{n}\right) - A_r(a)\right].$$

It is thus sufficient to show that

$$\sum_1^n \left[A_r\left(a + m\frac{\pi}{n}\right) - A_r(a)\right]$$

is uniformly bounded for all n—i.e., that $\sum_1^n A_r(x)$ is uniformly bounded.

If $f(x)$ is bounded and measurable, the first Cesàro mean is uniformly bounded for all x and n. For if the nth partial sum of this mean is denoted by σ_n, we have

$$\sigma_n(x) = \frac{1}{n\pi}\int_0^\pi f(x + 2t)\,\frac{\sin^2 nt}{\sin^2 t}\,dt$$

$$|\sigma_n(x)| \leqq \frac{B}{n\pi}\int_0^\pi \frac{\sin^2 nt}{\sin^2 t}\,dt = B, \quad (|f(x)| \leqq B).$$

From this the result follows. For if $f(x)$ is bounded and has Fourier coefficients $O(1/n)$, the partial sums of the Fourier development are uniformly bounded. In fact

$$s_n - \sigma_n = \frac{A_1 + 2A_2 + \cdots + nA_n}{n+1}$$

where s_n is the nth partial sum of $\sum_1^\infty An(x)$. But $nA_n = O(1)$, so it follows that the partial sums of the series are bounded uniformly. This completes the proof of Theorem 1.

As a particular example we note that the conditions of the theorem are satisfied for the function $\psi(n) = n(\log n)^\alpha$, $0 < \alpha \leqq 1$. For we have

$$\frac{d\psi}{dn} = \frac{\alpha}{(\log n)^{1+\alpha}} + (\log n)^\alpha > (\log n)^\alpha = \frac{\psi(n)}{n}.$$

Since the condition that $\psi(n) \geqq n$ was used in the proof of Theorem 1 only to show that the partial sums of the Fourier series of $f(x)$ were uniformly bounded, we may state the following theorem.

COROLLARY TO THEOREM 1. *If the Fourier coefficients of a bounded and measurable function $f(x)$ of period 2π are of the order $1/\psi(n)$ with the condition $d\psi(t)/dt > k\psi(t)/t$ satisfied for some $k>0$, and if the partial sums of the Fourier series of $f(x)$ are uniformly bounded, then $f(x)$ is of class $\psi(n)/\log n$.*

Following the terminology of Bray,[4] we shall say that a function $f(x)$ *is of class $\phi(n)$ and admits the constant K if K is such that the inequalities*

$$\left| \phi(n) \int_a^b f(x+k) \cos nx dx \right| \leqq K$$

$$\left| \phi(n) \int_a^b f(x+k) \sin nx dx \right| \leqq K$$

are satisfied for all $n \geqq 1$, k arbitrary, and $b-a \leqq 2\pi$.

LEMMA 3. *If $f(x)$ is of class $\phi(n)$, admitting the constant K, and if $|f(x)| \leqq M$, there exist positive constants A and B independent of $f(x)$ such that*

$$| S_n(x) | \leqq M\left(A \log \theta(n) + B \frac{n}{\phi(n)}\right) + \frac{2K}{\theta(n)}$$

where $S_n(x)$ is the nth partial sum of the Fourier series of $f(x)$ and $\theta(n)$ is any monotone increasing function such that $1 \leqq \theta(n) \leqq \phi(n)$.

In order to prove this we consider

$$S_n(x) = \frac{1}{\pi} \int_0^\pi [f(x+t)+f(x-t)] \frac{\sin (n+\frac{1}{2})t}{2 \sin \frac{1}{2}t} dt$$

$$= \frac{1}{\pi} \int_0^{\theta(n)/\phi(n)} + \int_{\theta(n)/\phi(n)}^{\pi} [f(x+t)+f(x-t)] \frac{\sin (n+\frac{1}{2})t}{2 \sin \frac{1}{2}t} dt.$$

[4] Loc. cit., p. 160.

Hence

$$(17)\quad \begin{aligned} | S_n(x) | &\leqq \frac{M}{\pi} \int_0^{\theta(n)/\phi(n)} \left| \frac{\sin (n+\frac{1}{2})t}{\sin \frac{1}{2}t} \right| dt \\ &+ \frac{1}{\pi} \left| \int_{\theta(n)/\phi(n)}^{\pi} f(x+t) + f(x-t)] \frac{\sin (n+\frac{1}{2})t}{2 \sin \frac{1}{2}t} dt. \right. \end{aligned}$$

The first integral in (17) does not exceed

$$\int_0^{1/\phi(n)} \left| \frac{\sin (n+\frac{1}{2})t}{\sin \frac{1}{2}t} \right| dt + \int_{1/\phi(n)}^{\theta(n)/\phi(n)} \frac{4}{t} dt$$

which is not greater than $(2n+1)/\phi(n)+4 \log \theta(n)$. Thus the first term of (17), is not greater than

$$\frac{M}{\pi}\left[\frac{2n+1}{\phi(n)} + 4 \log \theta(n)\right].$$

For the second integral we obtain, by using the second mean value theorem,

$$\begin{aligned} &\int_{\theta/\phi}^{\pi} [f(x+t) + f(x-t)] \frac{\sin (n+\frac{1}{2})t}{2 \sin \frac{1}{2}t} dt \\ &\qquad = \tfrac{1}{2} \csc \frac{\theta(n)}{2\phi(n)} \int_{\theta(n)/\phi(n)}^{t_n} [f(x+t) + f(x-t)] \sin (n+\tfrac{1}{2})t dt \end{aligned}$$

where

$$\frac{\theta(n)}{\phi(n)} \leqq t_n \leqq \pi.$$

But

$$\left| \int_a^b f(x+t) \sin nx dx \right| \leqq K/\phi(n).$$

Hence

$$\begin{aligned} &\int_{\theta(n)/\phi(n)}^{t_n} [f(x+t) + f(x-t)] \sin (n+\tfrac{1}{2})t dt \\ &\qquad\qquad \leqq \frac{2K}{\phi(n+\frac{1}{2})} \leqq \frac{2K}{\phi(n)}. \end{aligned}$$

Thus the second term of (17) does not exceed

$$\frac{1}{2\pi}\csc\frac{\theta(n)}{2\phi(n)}\,\frac{2K}{\phi(n)} \leqq \frac{1}{2\pi}\,\frac{4\phi(n)}{\theta(n)}\,\frac{2K}{\phi(n)} = \frac{4K}{\pi\theta(n)} \leqq \frac{2K}{\theta(n)}$$

and we get the desired result.

Lemma 4. *Under the hypotheses of Theorem 3 there exist positive constants A and B independent of $f(x)$ such that the remainder $R_n(x)=f(x)-S_n(x)$ satisfies the inequality*

$$|R_n| \leqq M\left(A\log\theta(n) + B\frac{n}{\phi(n)}\right) + \frac{2K}{\theta(n)}\cdot$$

For

$$\begin{aligned} |R_n| &= |S_n - f| \leqq |S_n| + |f| \\ &\leqq M\left[A\log\theta(n) + B'\frac{n}{\phi(n)}\right] + \frac{2K}{\theta(n)} + M; \end{aligned}$$

$$\begin{aligned} |R_n| &\leqq M\left[A\log\theta(n) + B'\frac{n}{\phi(n)} + 1\right] + \frac{2K}{\theta(n)} \\ &\leqq M\left[A\log\theta(n) + B\frac{n}{\phi(n)}\right] + \frac{2K}{\theta(n)} \end{aligned}$$

since

$$B'\frac{n}{\phi(n)} + 1 \leqq (B'+1)\frac{n}{\phi(n)} = B\frac{n}{\phi(n)}\cdot$$

Lemma 5. If $f(x)$ has first and second order derivatives $f'(x)$, $f''(x)$, and if $f''(x)$ is of class $\phi(n)$ admitting the constant K'', with $|f''(x)| \leqq M''$; then if $\phi'(n) < B'$ there exist positive constants A and B such that

$$|R_n| \leqq \frac{M''}{n^2}\left[A\log\theta(n) + B\frac{n}{\phi(n)}\right] + \frac{4K''}{n^2\theta(n)}$$

where $\theta(n)$ is any monotone increasing function such that

$$1 \leqq \theta(n) \leqq \phi(n); \qquad 1 \leqq \frac{\theta(n+1)}{\theta(n)} \leqq \frac{\phi(n+1)}{\phi(n)}\cdot$$

We let $S_n(x) = \sum_1^n A_k(x)$ and let the Fourier partial sum of $f''(x)$ be

$$\sigma_n = \sum_1^n B_k(x) = -\sum_1^n k^2(a_k \cos kx + b_k \sin kx)$$
$$= -\sum_1^n k^2 A_k(x).$$

We thus obtain

$$R_n = \sum_{n+1}^{\infty} A_k(x) = -\sum_{n+1}^{\infty} \frac{B_k}{k^2} = \sum_{n+1}^{\infty} \frac{1}{k^2}(\sigma_k - \sigma_{k-1}).$$

Hence

$$R_n = \frac{\sigma_n}{(n+1)^2} - \sum_{n+1}^{\infty} \sigma_k \left(\frac{1}{k^2} - \frac{1}{(k+1)^2}\right).$$

But

$$|\sigma_k| \leqq M''\left(A \log \theta(k) + B\frac{k}{\phi(k)}\right) + \frac{2K''}{\theta(k)}$$

from Lemma 3. Hence we have

$$\begin{aligned} |R_n| \leqq & \frac{M''}{(n+1)^2}\left[A \log \theta(n) + B\frac{n}{\phi(n)}\right] + \frac{2K''}{\theta(n)(n+1)^2} \\ & + 2K''\sum_{n+1}^{\infty} \frac{k}{\phi(k)}\left(\frac{1}{k^2} - \frac{1}{(k+1)^2}\right) \qquad (18) \\ & + AM''\sum_{n+1}^{\infty} \log \theta(k)\left(\frac{1}{k^2} - \frac{1}{(k+1)^2}\right). \end{aligned}$$

The fourth term of (18) does not exceed

$$\frac{2K''}{\theta(n+1)} \sum_{n+1}^{\infty} \left(\frac{1}{k^2} - \frac{1}{(k+1)^2}\right) \leqq \frac{2K''}{\theta(n+1)} \frac{1}{(n+1)^2}.$$

The fifth term is equal to

$$(19)\qquad AM''\left[\frac{\log\theta(n+1)}{(n+1)^2}+\sum_{n+1}^{\infty}\left\{\left(\log\frac{\theta(k+1)}{\theta(k)}\right)\frac{1}{(k+1)^2}\right\}\right].$$

The second term of (19) is not greater than

$$\sum_{n+1}^{\infty}\frac{1}{(k+1)^2}\log\frac{\theta(k+1)}{\theta(k)}\leqq\sum_{n+1}^{\infty}\frac{B'}{\phi(k)}\frac{1}{(k+1)^2}$$
$$\leqq\frac{B'}{\phi(n)}\sum_{n+1}^{\infty}\frac{1}{(k+1)^2}\leqq\frac{1}{n^2}B'\frac{n}{\phi(n)}$$

since

$$\log\frac{\phi(k+1)}{\phi(k)}\leqq\frac{B'}{\phi(k)}\cdot$$

This follows at once from the law of the mean and one of the hypotheses of the theorem.

Thus the fifth term of (18) does not exceed

$$(20)\qquad AM''\left(\frac{\log\theta(n+1)}{(n+1)^2}+\frac{1}{n^2}B'\frac{n}{\phi(n)}\right)\cdot$$

Since $\phi'(n)\leqq B$ it follows that

$$\phi(n+1)\leqq B'+\phi(n)\leqq 2\phi(n)$$

for n sufficiently large. Then $\theta(n+1)/\theta(n)\leqq 2$ and $\theta(n+1)\leqq 2\theta(n)$. Hence

$$\log\theta(n+1)\leqq\log\theta(n)+\log 2\leqq 2\log\theta(n)$$

and the expression (20) is not greater than

$$\frac{AM''}{n^2}\left(2\log\theta(n)+\frac{1}{n^2}B\frac{n}{\phi(n)}\right).$$

The third term of (19) does not exceed

$$\frac{BM''}{\phi(n)}\sum_{n+1}^{\infty}k\left(\frac{2k+1}{k(k+1)^2}\right)\leqq\frac{2BM''}{\phi(n)}\sum_{n+1}^{\infty}\frac{1}{k^2}$$
$$\leqq\frac{2BM''}{n\phi(n)}\leqq\frac{M''}{n^2}2B\frac{n}{\phi(n)}\cdot$$

By combining the above results we get

$$| R_n | \leqq \frac{M}{n^2}\left(3A \log \theta(n) + 3B \frac{n}{\phi(n)}\right) + \frac{4K''}{n^2\theta(n)}$$

which has the desired form.

THEOREM 2. *If $f(x)$ is of class $\phi(n)$ with $\phi'(n) < B'$ and if $f(x)$ admits the constant K and is continuous with modulus of continuity $\omega(\delta)$, then there exist positive constants A, B, and C independent of $f(x)$ such that*

$$(21) \qquad | R_n | \leqq \omega\left(\frac{1}{n}\right)\left[A \log \theta(n) + B \frac{n}{\phi(n)}\right] + \frac{CK}{\theta(n)}$$

where $\theta(n)$ is monotone increasing and such that

$$1 \leqq \theta(n) \leqq \phi(n); \qquad 1 \leqq \frac{\theta(n+1)}{\theta(n)} \leqq \frac{\phi(n+1)}{\phi(n)}.$$

In order to prove this theorem we employ the averaging function[5]

$$f_\mu(x) = \frac{1}{\mu^2}\int_0^\mu dt \int_0^\mu f(x+t+t')dt'.$$

This function has the following properties, as may be easily verified:

(i) If $f(x)$ is continuous,

$$\frac{d}{dx} f_\mu(x) = \frac{1}{\mu^2}\int_0^\mu [f(x+\mu+t) - f(x+t)]dt.$$

(ii) If $f(x)$ is continuous,

$$\frac{d^2}{dx^2} f_\mu(x) = \frac{1}{\mu^2}[f(x+2\mu) - 2f(x+\mu) + f(x)].$$

(iii) If $f(x)$ is continuous, of class $\phi(n)$, admits the constant K, and has modulus of continuity $\omega(\delta)$, then $f_\mu(x)$ is of class $\phi(n)$ and admits the constant K.

[5] See Bray, Bull. Am. Math. Soc. Vol. 29, No. 6.

(iv) Under the conditions of (iii) $f_\mu''(x)$ is of class $\phi(n)$ and admits the constant $4K/\mu^2$.

(v) Under the conditions of (iii)

$$|f_\mu''(x)| \leqq \frac{2\omega(\mu)}{\mu^2}.$$

(vi) Under the conditions of (iii)

$$|f_\mu(x) - f(x)| \leqq \omega(2\mu).$$

In order to prove the theorem we write

$$f(x) = (f(x) - f_\mu(x)) + f_\mu(x).$$

We then have $R_n = R_n' + R''$ where R_n, R_n', and R'' are the Fourier remainders of order n of $f(x)$, $f(x) - f_\mu(x)$, and $f_\mu(x)$, respectively. It is clear from (iii) that $f(x) - f_\mu(x)$ is of class $\phi(n)$ and admits the constant $2K$. From Lemma 4, using the constants of Lemma 5, we have

$$|R_n'| \leqq \omega(2\mu)\left(A \log \theta(n) + B\frac{n}{\phi(n)}\right) + \frac{4K}{\theta(n)}.$$

Since $f_\mu''(x)$ is of class $\phi(n)$ and admits the constant $4K/\mu^2$ we have from Lemma 5

$$|R_n''| \leqq \frac{2\omega(\mu)}{\mu^2 n^2}\left[A \log \theta(n) + B\frac{n}{\phi(n)}\right] + \frac{16K}{n^2\mu^2\theta(n)}.$$

If we put $\mu = 1/2n$, we then have

$$\begin{aligned} |R_n| &= |R_n'| + |R_n''| \\ &\leqq 9\omega\left(\frac{1}{n}\right)\left[A \log \theta(n) + B\frac{n}{\phi(n)}\right] + \frac{68K}{\theta(n)}, \end{aligned}$$

since $\omega(1/2n) \leqq \omega(1/n)$.

Thus there are positive constants A, B, and C such that

$$|R_n| \leqq \omega\left(\frac{1}{n}\right)\left(A \log \theta(n) + B\frac{n}{\phi(n)}\right) + \frac{CK}{\theta(n)},$$

and the theorem is proved.

From this last theorem we see that if $\phi(n)$ is such that $\omega(1/n)n/\phi(n)\to 0$ as $n\to\infty$ and if $\theta(n)$ is so chosen that $\omega(1/n)\log\theta(n)\to 0$ while $\theta(n)\to\infty$, then the Fourier series of $f(x)$ converges uniformly to $f(x)$.

3. EXAMPLE. *A Function of Class $\phi(n)=n/(\log n)^{1/3}$ having a Uniformly Convergent Fourier Series*:

The function we shall consider is defined as follows. We let

$$f(x) = k(x)\sum_{i=1}^{\infty}\lambda_i(x) \qquad \text{for } x \neq 0$$
$$= 0 \qquad \text{for } x = 0$$

where

$$\lambda_i(x) = \sin n_i x \qquad \text{for } \alpha_i \leqq x \leqq \beta_i$$
$$= 0 \qquad \text{otherwise.}$$

We set

$$\mu(x) = 1\bigg/\left(\log\frac{1}{x}\right)^{2/3}, \qquad 0 \leqq x \leqq \pi/4$$
$$= \mu(\pi/4), \qquad \pi/4 \leqq x,$$

and let

$$k(x) = \mu(\alpha_i) \qquad \text{for } \alpha_i \leqq x \leqq \beta_i$$
$$= 0 \qquad \text{otherwise,}$$

$n_i = 2^i 2^{2^i}$, $\alpha_i = 4\pi/2^{2^i}$. β_i is a number such that $\beta_i > \alpha_i$ and $\beta_i \sim \alpha_i$. For convenience we take $\beta_i = 2\alpha_i$.

By the notation $\alpha\sim\beta$ we shall mean that α/β is bounded from zero and infinity. We say that α and β have the same order, and we shall also write $\alpha/\beta\sim 1$.

It is clear that the function $f(x)$ defined above is continuous in $0\leqq x\leqq 2\pi$. The graph of the function is illustrated in the accompanying figure.

If the function $\phi(n)$ is the function $n/(\log n)^{1/3}$, we shall show first that $f(x)$ is at least of this class. We have

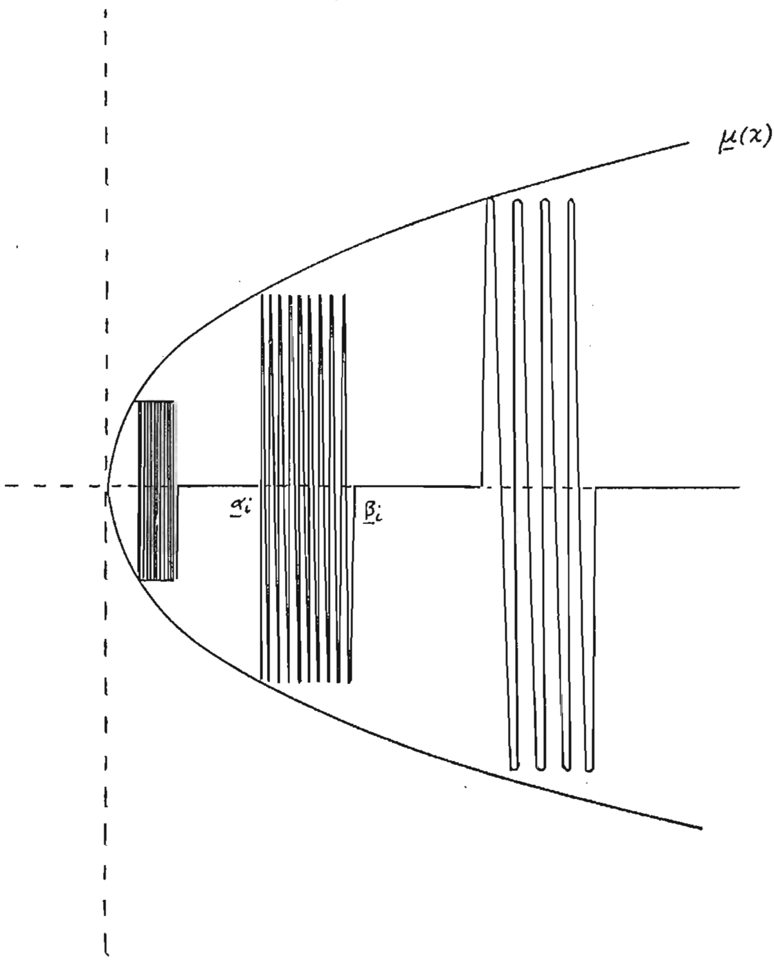
$$\left|\int_{\alpha_i}^{\beta_i} f(x)\sin n_i x\,dx\right| = \left|\int_{\alpha_i}^{\beta_i} k(x)\sin^2 n_i x\,dx\right| \sim k(\alpha_i)\alpha_i$$

since $\beta_i - \alpha_i \sim \alpha_i$.

Now

$$k(\alpha_i)\alpha_i = \frac{\alpha_i}{\left(\log \frac{1}{\alpha_i}\right)^{2/3}} \sim \frac{1}{2^{2i}(\log 2^{2i})^{2/3}}$$

$$\sim \frac{2^i}{n_i(\log n_i)^{2/3}} \sim \frac{(\log n_i)^{1/3}}{n_i}$$



since $\log n_i \sim i+2^i \sim 2^i$. This is the desired result.

We now show that $f(x)$ is at most of class $\phi(n)=n/(\log n)^{1/3}$. We consider $\int_a^b f(x) \sin nx dx$. The number n may be expressed in the form $n=np+t$ where $0 \leqq t < n_{p+1}-n_p$. Then $n_p \leqq n < n_{p+1}$. We thus have

$$\int_a^b f(x) \sin nx dx = \int_a^b f(x) \sin (n_p + t) x dx.$$

This integral may be expressed as

$$\int_0^b - \int_0^a f(x) \sin (n_p + t) x dx$$

and it is thus sufficient to consider integrals of this type. We write

$$\begin{aligned} \int_0^b f(x) &\sin (n_p + t) x dx \\ &= \sum_{i=m}^{\infty} \int_{\alpha_i}^{\gamma_i} f(x) \sin (n_p + t) x dx \\ &= \sum_{i=m}^{\infty} \int_{\alpha_i}^{\gamma_i} k(x) \sin n_i x \sin (n_p + t) x dx \end{aligned} \tag{22}$$

where $\alpha_i < \gamma_i \leqq \beta_i$; $m \geqq 1$. The right member of (22) may be written as

$$\sum_{i=m}^{p-1} + \sum_{i=p}^{\infty} \int_{\alpha_i}^{\gamma_i} k(x) \sin n_i x \sin (n_p + t) x dx. \tag{23}$$

The second sum of (23) may be written as

$$\begin{aligned} \int_{\alpha_p}^{\gamma_p} & k(x) \sin n_p x \sin (n_p + t) x dx \\ &+ \sum_{i=p+1}^{\infty} \int_{\alpha_i}^{\gamma_i} k(x) \sin n_i x \sin (n_p + t) x dx. \end{aligned} \tag{24}$$

In this sum we have

$$\sum_{i=p+1}^{\infty} \left| \int_{\alpha_i}^{\gamma_i} k(x) \sin n_i x \sin (n_p + t) x dx \right| \sim \sum_{p+1}^{\infty} k(\alpha_i)\alpha_i,$$

and this does not exceed

$$k(\alpha_{p+1}) \sum_{p+1}^{\infty} \alpha_i \sim k(\alpha_{p+1})\alpha_{p+1}.$$

But

$$k(\alpha_{p+1})\alpha_{p+1} \sim \frac{(\log n_{p+1})^{1/3}}{n_{p+1}},$$

which does not exceed a quantity of the order

$$\frac{\log (n_p + t)^{1/3}}{n_p + t} = \frac{(\log n)^{1/3}}{n}.$$

In considering the first term of (24) we distinguish two cases:

Case I. $t < n_p$. We write

$$\left| \int_{\alpha_p}^{\gamma_p} k(x) \sin n_p x \sin (n_p + t) x dx \right| \sim k(\alpha_p)\alpha_p.$$

Hence

$$k(\alpha_p)\alpha_p \sim \frac{(\log n_p)^{1/3}}{n_p} \sim \frac{(\log (n_p + t))^{1/3}}{n_p + t} = \frac{(\log n)^{1/3}}{n}.$$

For, since $t < n_p$, $n_p + t \sim n_p$.

Case II. $t \geqq n_p$. In this case we write

$$\begin{aligned} &\left| \int_{\alpha_p}^{\gamma_p} k(x) \sin n_p x \sin (n_p + t) x dx \right| \\ (25) \qquad &= \left| \int_{\alpha_p}^{\gamma_p} k(x) \cos t x dx - \int_{\alpha_p}^{\gamma_p} k(x) \cos (2n_p + t) x dx \right|. \end{aligned}$$

For the second term in the right member of (25) we have

$$\begin{aligned} \left| \int_{\alpha_p}^{\gamma_p} k(x) \cos (2n_p + t) x dx \right| &\sim \frac{k(\alpha_p)}{2n_p + t} \sim \frac{k(\alpha_p + t)}{n_p + t} \\ &\leqq \frac{(\log (n_p + t))^{1/3}}{n_p + t} \sim \frac{(\log n)^{1/3}}{n} \end{aligned}$$

$$\frac{k(\alpha_p)}{n_p} < k(\alpha_p)\alpha_p.$$

The first integral in (25) gives us

$$\left|\int_{\alpha_p}^{\gamma_p} k(x) \cos txdx\right| = O\left(\frac{k(\gamma_p)}{t}\right) \sim \frac{k(\alpha_p)}{t}.$$

But here we have $t \geqq n_p$, so that $t \sim n_p+t$. In fact we may say that $t \geqq (n_p+1)/2$. Thus, from the above relation,

$$\left|\int_{\alpha_p}^{\gamma_p} k(x) \cos txdx\right| \leqq \frac{2k(\alpha_p)}{n_p+t} \sim \frac{k(\alpha_p+t)}{n_p+t}$$
$$\sim \frac{(\log (n_p+t))^{1/3}}{n_p+t} = \frac{(\log n)^{1/3}}{n}.$$

Thus far we have the desired result for the infinite sum in (23). We now prove it for the finite part. We have

$$\begin{aligned} &\sum_{i=m}^{p-1} \left|\int_{\alpha_i}^{\gamma_i} k(x) \sin n_i x \sin (n_p+t)xdx\right| \\ (26) \qquad &\leqq \sum_{i=1}^{p-1} \left|\int_{\alpha_i}^{\gamma_i} k(x) \cos (n_p+t-n_i)xdx\right| \\ &\quad + \sum_{1}^{p-1} \left|\int_{\alpha_i}^{\gamma_i} k(x) \cos (n_p+t+n_i)xdx\right|. \end{aligned}$$

The function $k(x)$ plays no part in determining the order here since its greatest value is for the index $i=1$. The first sum in the right member of (26) can then be taken to be

$$\sum_{i=1}^{p-1} \left|\int_{\alpha_i}^{\gamma_i} \cos (n_p+t-n_i)xdx\right| \sim \sum_{i=1}^{p-1} \frac{1}{n_p+t-n_i}.$$

The greatest value of $1/(n_p+t-n_i)$ is attained for $i=p-1$, namely $1/(n_p+t-n_{p-1})$. But

$$\frac{1}{n_p+t-n_{p-1}} \sim \frac{1}{n_p+t}.$$

Hence

$$\sum_{i=1}^{p-1} \frac{1}{n_p+t-n_i} \leqq \frac{p}{n_p+t},$$

and since $p \sim \log \log n_p$, it follows that

$$\sum_{i=m}^{p-1} \left| \int_{\alpha_p}^{\gamma_i} \cos (n_p + t - n_i) x dx \right| \leqq \frac{\log \log n_p}{n_p + t}$$
$$< \frac{(\log (n_p + t))^{1/3}}{n_p + t} = \frac{(\log n)^{1/3}}{n}.$$

In a similar manner we see that the second sum of (26) can be considered in the form

$$\sum_{i=1}^{p-1} \left| \int_{\alpha_i}^{\gamma_i} \cos (n_p + t + n_i) x dx \right| \sim \sum_{i=1}^{p-1} \frac{1}{n_p + t + n_i}$$
$$\leqq \sum_{i=1}^{p-1} \frac{1}{n_p + t} \leqq \frac{p}{n_p + t}.$$

The rest of the argument is as before. The proof is now complete, and we see that $f(x)$ is indeed of class

$$\phi(n) = n/(\log n)^{1/3}.$$

We wish now to determine the modulus of continuity of $f(x)$. We shall consider the function in the interval $[\alpha_i, \beta_i]$. Since we consider max $|f(x_2) - f(x_1)|$, $|x_2 - x_1| < \delta$, it is sufficient to consider only right hand differences. We may write

$$\left| \frac{\Delta f}{\Delta x} \right| < K \left| f'(x) \right|$$

where K is a positive constant. We have

$$\begin{aligned} f'(x) &= k(x) n_i \cos n_i x + k'(x) \sin n_i x \\ &= k(x) n_i \cos n_i x \end{aligned}$$

and

$$\left| f'(x) \right| \leqq \left| k(x) n_i \right| = \mu(\alpha_i) n_i.$$

It is convenient to consider three cases here.

Case I. We consider two points x_1 and $x_2 = x_1 + \delta$ of $[\alpha_i, \beta_i]$ with δ less than a half period of $f(x)$. Then we have $|f(x_2) - f(x_1)| < K n_i \mu(\alpha_i) \delta$. For δ equal to a half period this is of the order $\mu(\alpha_i)$ since then $\delta \sim 1/n_i$. But then $\mu(\alpha_i) \sim \mu(1/n_i) \sim \mu(\delta)$. Since for a half period $Kn_i\mu(\alpha_i)\delta \sim \mu(\delta)$ it follows that $Kn_i\mu(\alpha_i)\delta = C\mu(\delta)$ and $|\Delta f| < C\mu(\delta)$.

Case II. If x_1 and x_2 are in $[\alpha_i, \beta_i]$ and δ is greater than a half period, then, since $\mu(\delta)$ is an increasing function and Δf can be no greater than the maximum in a half period, the result of Case I holds *a fortiori.*

Case III. If x_1 is in the interval $[\alpha_i, \beta_i]$ and $\delta > \alpha_i = \beta_i - \alpha_i$, then

$$| f(x_2) - f(x_1) | = | f(x_1 + \delta) - f(x_1) | < 2\mu(x_1 + \delta) \sim C\mu(\delta)$$

since $x_1 \sim \alpha_1$ and $\delta > \alpha_1$ so that $x_1 + \delta \sim \delta$.

If we take the point x_1 at the origin, then, since $f(0) = 0$, we have $\max |f(x_2) - f(x_1)| = \max |f(x_2)|$ which is $\max |f(x_1+\delta)| = \max |f(\delta)| = \mu(\delta)$.

Thus the modulus of continuity of the function $f(x)$ is of the order

$$\mu(\delta) \sim \frac{1}{\left(\log \frac{1}{\delta}\right)^{2/3}}.$$

From Theorem 6 we have

$$| R_n | \leqq \omega\left(\frac{1}{n}\right)\left[A \log \theta(n) + B \frac{n}{\phi(n)}\right] + \frac{CK}{\theta(n)}$$

with

$$1 \leqq \theta(n) \leqq \phi(n); \qquad 1 \leqq \frac{\theta(n+1)}{\theta(n)} \leqq \frac{\phi(n+1)}{\phi(n)}.$$

For the example

$$\phi(n) = \frac{n}{(\log n)^{1/3}}, \qquad \omega(\delta) = \frac{1}{\left(\log \frac{1}{\delta}\right)^{2/3}}.$$

We may take $\theta(n) = \log n$. Then we have

$$1 \leqq \log n \leqq \frac{n}{(\log n)^{1/3}}$$

and

$$1 \leqq \frac{\log (n+1)}{\log n} \leqq \frac{n+1}{n}\left(\frac{\log n}{\log (n+1)}\right)^{1/3}.$$

Hence

$$|R_n| \leqq \frac{1}{(\log n)^{2/3}}\left[A \log \log n + B(\log n)^{1/3}\right] + \frac{CK}{\log n}$$
$$\leqq \frac{A \log \log n}{(\log n)^{2/3}} + \frac{B}{(\log n)^{1/3}} + \frac{CK}{\log n},$$

and this approaches zero as $n \rightarrow \infty$. It follows that the Fourier series of $f(x)$ converges uniformly.

A more general example is obtained if we take

$$\mu(x) = \frac{1}{\left(\log \dfrac{1}{x}\right)^{s}}, \qquad \phi(n) = \frac{n}{(\log n)^{1-s}}$$

with $s > 1/2$.

JOHN P. NASH